# GEOMETRIC INVARIANTS OF NUMERICAL SEMIGROUPS

MAKSYM FEDORCHUK, ANDREW FERDOWSIAN, JIAN ZHOU

Abstract. A natural invariant of a unibranch curve singularity is the numerical semigroup of its valuations. In the case when the curve singularity admits a $\mathbb{G}G_m$-action, this semigroup also determines the singularity uniquely. The paper [AFS14] proposed a rational-valued function on curve singularities with $\mathbb{G}G_m$-action that leads to an ordering of singularities according to their geometric complexity. We explore this function and give a classification of those numerical semigroups for which the values of this function are above a certain threshold.

## 1. Introduction

A *numerical semigroup* $N$ is a subset of non-negative integers $\mathbb{N}_0$ that is closed under addition, contains 0, and such that its complement $\mathbb{N}_0 \setminus N$ has finite cardinality, which is called *the genus of* $N$. Elements of $\mathbb{N}_0 \setminus N$ are called *the gaps of* $N$; these are denoted by $b_1 < \cdots < b_g$. We say that a semigroup $N$ is *symmetric* if the following holds:

$$i \in \mathbb{N}_0 \text{ is a gap if and only if } b_g - i \text{ is not a gap.} \tag{1.1}$$

Given integers $n_1 < \cdots < n_k$, we denote by $\langle n_1, \dots, n_k \rangle$ the semigroup formed by all linear combinations $a_1 n_1 + \cdots + a_k n_k$, where $a_i \in \mathbb{N}_0$.

We say that $n_1 < \ldots < n_k$ are *minimal generators* of a semigroup $N$ if $N = \langle n_1, \dots, n_k \rangle$ and, for every $2 \le i \le k$, we have that $n_i \notin \langle n_1, \dots, n_{i-1} \rangle$.

Given a numerical semigroup $N = \{0, n_1, n_2, \dots, \} = \mathbb{N} \setminus \{b_1, \dots, b_g\}$, we call $n_1$, the first non-zero element of $N$, *the multiplicity of* $N$. The sum

$$w(N) := \sum_{i=1}^{g} (b_i - i) = \sum_{i=1}^{g} b_i - \frac{g(g+1)}{2}$$

is called *the Weierstrass weight* of $N$. In this paper, we study a closely related function

$$R(N) := \frac{(2g-1)^2}{\sum_{i=1}^{g} b_i} \tag{1.2}$$

that was introduced in the work of Alper-Fedorchuk-Smyth [AFS14], and which is expected to measure the geometric complexity of the singularity. We partially justify this expectation by classifying all symmetric semigroups which satisfy the inequality

$$R(N) = \frac{(2g-1)^2}{\sum_{i=1}^{g} b_i} \leq 4, \tag{1.3}$$

and showing that apart from two exceptional cases, all such singularities are planar singularities of type A.

1.1. **Outline of the paper:** In order to determine which semigroups satisfy the equation, we first find an upper bound for the value of each gap. In turn this enables us to find a relation between the value of a gap and the value of the preceding gap. Using these facts we show there exists an upper bound for the sum of the gaps of a semigroup, dependent upon the minimum generator of the semigroup. This value for the upper bound implies that only semigroups of minimal generator 2, 3, or 4 can satisfy the inequality. Lastly, we determine exactly which semigroups satisfy the inequality.

## 2. Main result

**Definition 2.1.** One class of numerical semigroups is given by hyperelliptic semigroups, which are defined to be semigroups generated by two elements one of which is 2 and the other is an odd integer greater than 1. Hence every hyperelliptic semigroup can be written as $\langle 2, 2k+1\rangle$, where $k \geq 1$ is an integer.

*Remark* 2.2. Every hyperelliptic semigroup is symmetric. Indeed, for $N = \langle 2, 2k+1\rangle$, the gaps are $1, 3, \dots, 2k-1$ and so the symmetry condition (1.1) is clearly satisfied.

The main result of this paper is the following:

**Theorem 2.3.** *The numerical semigroups that satisfy Inequality* (1.3) *are as follows:*

1. *All hyperelliptic semigroups.*
2. *The symmetric semigroups* $\langle 3,4\rangle, \langle 3,5\rangle, \langle 3,7\rangle$*, and* $\langle 4,5,6\rangle$*.*
3. *The non-symmetric semigroups* $\langle 3,4,5\rangle$ *and* $\langle 3,5,7\rangle$*.*

We begin with a simple

**Lemma 2.4.**

*(1) A numerical semigroup satisfies* (1.3) *if and only if it satisfies*

$$\sum_{i=1}^{g} b_i > g(g-1). \tag{2.1}$$

*(2) Every hyperelliptic semigroup satisfies Inequality* (1.3).

*Proof.* (1) is immediate and (2) follows from (1) by noting that for a hyperelliptic semigroup $N = \langle 2, 2k+1 \rangle$, we have $g = k$ and

$$\sum_{i=1}^{k} b_i = 1 + 3 + \cdots + (2k-1) = k^2 > k^2 - k.$$

□

It remains to establish Theorem 2.3 for non-hyperelliptic semigroups. In the process, we correct an erroneous claim made in [AFS14, Remark 3.7] regarding the classification of symmetric semigroups satisfying Inequality 2.1. Namely, we note that [AFS14, Remark 3.7] missed the symmetric semigroup $\langle 4, 5, 6 \rangle$ with the gap sequence $\{1, 2, 3, 7\}$ and genus 4.

The following is the main observation used in the proof of Theorem 2.3.

**Lemma 2.5.** *Suppose $N$ is a numerical semigroup of genus $g$ and $\{b_1, \ldots, b_g\}$ are gaps of $N$. Then $b_i \leq 2i - 1$.*

*Proof.* Note that because $b_i$ is a gap, every pair $(k, b_i - k)$, for $1 \leq k \leq \lfloor b_i/2 \rfloor$, must have at least one gap. Indeed, if neither $k$ nor $b_i - k$ is a gap, then $k, b_i - k \in N$ implies $b_i = k + (b_i - k)$ is also in $N$, which is not the case.

Suppose $b_i \geq 2i$. Then there exist at least $i$ distinct pairs $(k, b_i - k)$ with $1 \leq k \leq \lfloor b_i/2 \rfloor$. Since each such pair must contain at least one gap, there must be at least $i$ gaps before $b_i$. This contradiction leads us to conclude that $b_i \leq 2i - 1$. □

**Definition 2.6.** When the $i^{th}$ gap satisfies $b_i = 2i - 1$, we will refer to it as a *extremal* gap. When $b_i = 2i - 2$, we will refer to it as a *sub-extremal* gap.

Note that $b_1$ is always extremal because $b_1 = 1$ for every numerical semigroup of genus $g \geq 1$.

**Lemma 2.7.** *$b_g$ is always extremal in a symmetric semigroup.*

*Proof.* Each pair $(b_g - k, k)$, with $k \leq b_g - 1$, contains exactly one gap and one non-gap by the definition of a symmetric semigroup. Hence $b_g$ is odd. Because the number of gaps less than $b_g$ is precisely $g - 1$, it then follows that $\lfloor b_g/2 \rfloor = g - 1$. □

In what follows, we denote by $n$ the minimal generator of $N$.

**Lemma 2.8.** *Suppose $b_i$ is extremal and $i \geq 2$, then necessarily $b_{i-1} = b_i - n$. Suppose $b_i$ is sub-extremal and $i \geq 3$, then*

$$b_{i-1} = \begin{cases} b_i - n, & \text{if } n \neq i, \\ b_i - n + 1, & \text{if } n = i. \end{cases}$$

*Proof of lemma:* Following the proof of Lemma 2.5, each pair of integers $(k, b_i - k)$, where $1 \leq k \leq \lfloor b_i/2 \rfloor$, must contain at least one gap. In fact, we claim that every such pair contains exactly one gap. Indeed, by the assumption $b_i = 2i - 1$ or $b_i = 2i - 2$, and so $i - 1$ such pairs exist before $b_i$. If any of these pairs had two distinct gaps, then there would exist at least $i$ gaps before $b_i$ resulting in a contradiction.

Since $n$ is the minimal generator of $N$, it follows that $1, 2, \ldots, n-1$ are all gaps. If every pair $(k, b_i - k)$ for $1 \leq k \leq n$ contains distinct numbers, then given $1, 2, \ldots, n-1$ are gaps it must be the case that $b_i - 1, b_i - 2, \ldots, b_i - (n - 1)$ are all non-gaps, and $b_i - n$ must be a gap. It follows that $b_{i-1} = b_i - n$.

Suppose now that $k = b_i - k$ for some $k \leq n$. Then $b_i$ is even and we must have $k = i - 1$. Then $i - 1$ is a gap and since $n$ is not a gap, we must have $n > i - 1$. Suppose now $n > i - 1$. Then $b_j = j$ for every $j \leq i - 1$. Since $b_i = 2i - 2 > i$, we must have $n = i$. In particular, $b_{i-1} = i - 1 = b_i - n + 1$. □

**Corollary 2.9.** *Suppose $n \geq 4$. If $b_i$ is an extremal or subextremal gap for some $i \geq 3$, then $b_{i-1}$ is neither extremal nor subextremal. In particular, $i \geq 4$ a posteriori.*

*Proof.* For any $i \geq 3$, if $b_i$ is extremal then $b_{i-1} = b_i - n = 2i - 1 - n = 2(i-1) - n + 1 < 2(i-1) - 2$, the final step is simply due to $n > 4$.

Similarly, if $b_i$ is sub-extremal, then $b_{i-1} = 2i - 2 - n + 1 = 2(i-1) - n + 1 < 2(i-1) - 2$, and again $b_{i-1}$ is neither extremal nor sub-extremal. □

**Lemma 2.10.** *Suppose $N$ is a non-hyperelliptic semigroup such that $b_1$ and $b_2$ are its only extremal or sub-extremal gaps. Then*

$$\sum_{i=1}^{g} b_i \leq g^2 - 2g + 3.$$

*For such a semigroup, Inequality (2.1) is not satisfied as long as $g \geq 3$ or $n \geq 4$.*

*Proof.* By assumption $b_i \leq 2i-3$ for every $3 \leq i \leq g$. We obtain the following inequalities:

$$\begin{aligned}
\sum_{i=1}^{g} b_i &= 1+2+\sum_{i=3}^{g} b_i \\
&\leq 1+2+\sum_{i=3}^{g}(2i-3) \\
&\leq 1+2+2\left(\frac{g(g-1)}{2}-2-1\right)-3(g-2) \\
&\leq 1+2+g^2+g-6-3g+6 \\
&\leq g^2-2g+3.
\end{aligned}$$

Hence $N$ can satisfy Inequality (2.1) only if $g \leq 2$. To see the last statement of the lemma, observe that if $n$ is the minimal generator of $N$, then 1, 2, and 3 are gaps and so $g \geq 3$. $\square$

**Proposition 2.11.** *Suppose $n \geq 4$, and $N$ has $k \geq 1$ extremal or sub-extremal gaps $b_{j_1}, \dots b_{j_k}$ with $j_1, \dots, j_k \geq 3$. Then*

$$\sum_{i=1}^{g} b_i \leq g^2-2g+3-(n-6)k. \tag{2.2}$$

*Moreover, the above inequality is strict if any of the gaps $b_{j_1}, \dots b_{j_k}$ is sub-extremal.*

*Proof.* Let us break the gaps $\{b_1, \dots, b_g\}$ into 3 sets:

- $A = \{b_1, b_2\}$, where necessarily $b_1 = 1$ and $b_2 = 2$,
- $B = \{b_{j_1-1}, b_{j_1}, b_{j_2-1}, b_{j_2}, \dots, b_{j_k-1}, b_{j_k}\}$, and
- $C = G - A - B$ (that is, $C$ consists of non extremal and non sub-extremal gaps).

By Corollary 2.9, $b_{j_1-1}, b_{j_2-1}, \dots, b_{j_k-1}$ are neither extremal nor subextremal. Hence $B$ contains precisely $2k$ elements. Note $b_3 = 3$ is not extremal or sub-extremal, which implies that $j_1 \geq 4$, and so $A$, $B$, and $C$ are disjoint sets. Since $C$ cannot have a negative number of elements, we must have $g-2k-2 \geq 0$, or $k \leq \frac{g-2}{2}$. We now proceed to estimate the sum of gaps.

First, $b_1 + b_2 = 3$. Second, for every pair $\{j_r - 1, j_r\}$, where $1 \leq r \leq k$, we have by Lemma 2.8 if $b_{j_r}$ is extremal,

$$b_{j_r-1} + b_{j_r} \leq (2j_r - 1 - n) + (2j_r - 1) = 4j_r - n - 2$$

and if $b_{j_r}$ is sub-extremal.

$$b_{j_r-1} + b_{j_r} \leq (2j_r - 2 - n + 1) + (2j_r - 2) = 4j_r - n - 3$$

In either case, we have

$$b_{j_r-1} + b_{j_r} \leq 4j_r - n - 2,$$

with strict inequality if $b_{j_r}$ is sub-extremal.

Finally,

$$\sum_{b_i \in C} b_i \leq \sum_{b_i \in C} (2i - 3)$$

because gaps in $C$ are neither extremal nor sub-extremal.

Putting this together, we obtain:

$$\begin{aligned}\sum_{i=1}^{g} (b_i - (2i-1)) &= \sum_{b_i \in A} (b_i - (2i-1)) + \sum_{b_i \in B} (b_i - (2i-1)) + \sum_{b_i \in C} (b_i - (2i-1)) \\ &\leq (0-1) + \sum_{r=1}^{k} (2-n) + \sum_{b_i \in C} (-2) \\ &\leq -1 + \sum_{r=1}^{k} (2-n) + (g - 2k - 2)(-2) \\ &\leq 3 - 2g + (6-n)k.\end{aligned}$$

The claim follows using the well-known formula $\sum_{i=1}^{g} (2i-1) = g^2$. □

**Corollary 2.12.** *Suppose $N$ is a numerical semigroup with the minimal generator $n > 4$. Then $N$ does not satisfy Inequality (2.1).*

*Proof.* By Proposition 2.11 we know $\sum b_i \leq g^2 - 2g + 3 - (n-6)k$. We aim to show $g^2 - 2g + 3 - (n-6)k \leq g(g-1)$, or equivalently,

$$-g + 3 - (n-6)k \leq 0 \tag{2.3}$$

We break this into two cases, if $n \geq 6$ then it follows that $1, 2, \ldots, 5$ are gaps and $g \geq 3$ therefore the inequality is satisfied as $(n-6) \leq 0, g \geq 3$.

If $n = 5$ then we show $-g + 3 + k \leq 0$. But as shown in Proposition 2.11 $k \leq \frac{g-2}{2}$, so we consider $-g + 3 + \frac{g-2}{2}$. Then, $-2g + 6 + g - 2 \leq 0$, which in turn implies $-g + 4 \leq 0$, however if $n = 5$ then $g \geq 4$ proving the corollary. □

**Corollary 2.13.** *Suppose $N$ is a numerical semigroup with the minimal generator $n = 4$. Then $N$ satisfies Inequality (2.1) if and only if $N = \langle 4, 5, 6 \rangle$.*

*Proof.* By Equation 2.3 we find that in order for the inequality to be satisfied $-g+3+2k>0$, when $k$ takes its maximum value we find that $k=\frac{g-2}{2}$, leaving the inequality as $-g+3+g-2>0$, or $1>0$.

Next we note that if the inequality in Proposition 2.11 is strict, this inequality cannot be satisfied. This in turn implies that the gaps pairs must be all extremal after $1,2$. If any were not extremal or sub-extremal then $k$ would not be maximized, additionally if any were sub-extremal then the inequality would be strict.

However, $4,5,6$ are all not gaps, as $1,2,3$ make up the first three gaps, then $b_4=7$ as it must be extremal. But if $4,5,6$ are not gaps this uniquely determines the semigroup implying that there exists a unique semigroup with minimal generator $n=4$ which satisfies the inequality. □

Summarizing, we can conclude that numerical semigroups with $n>4$ cannot satisfy Inequality (2.1) and the only semigroup with $n=4$ satisfying Inequality (2.1) is $N=\langle 4,5,6\rangle$. To finish the proof of Theorem 2.3, it remains to only discuss the cases $n=3$.

**The case of $n=3$.** Since the minimal generators of a numerical semigroup have different residue classes modulo $n$, any semigroup with $n=3$ would have at most 3 minimal generators. Therefore, we break our analysis into two cases: semigroups in the form $\langle 3,m\rangle$ or $\langle 3,m,l\rangle$.

**Proposition 2.14.** *Suppose $N=\langle 3,m\rangle$, where $3<m$, satisfies Inequality (1.3), then $N=\langle 3,4\rangle$, $\langle 3,5\rangle$, or $\langle 3,7\rangle$.*

*Proof.* Numerical semigroups of two generators are well-studied. It follows from the formula (32) in [Rød94] that

(a) $g=m-1$.

(b) $\displaystyle\sum_{i=1}^{g} b_i=\frac{5m^2-9m+4}{6}$.

Substitute into (2.1) yields $m^2-9m+8<0$, which holds if and only if $3<m<8$. Using the condition $\gcd(3,m)=1$, we obtain $m\in\{4,5,7\}$. □

**Proposition 2.15.** *Suppose $N=\langle 3,m,l\rangle$, where $3<m<l$, satisfies Inequality (1.3), then $N=\langle 3,4,5\rangle$ or $\langle 3,5,7\rangle$.*

*Proof.* Assume $m\equiv 1 \pmod 3$ and $l\equiv 2 \pmod 3$. Then

$$\mathbb{N}_0\setminus\langle 3,m,l\rangle=\{1,4,\dots,m-3\}\cup\{2,5,\dots,l-3\}$$

consists of two arithmetic sequences. We calculate

(a) $g = \dfrac{m+l-3}{3}$.
(b) $\displaystyle\sum_{i=1}^{g} b_i = \frac{(m-1)(m-2)+(l-1)(l-2)}{6}$.

If we assume that $m \equiv 2 \pmod 3$ and $l \equiv 1 \pmod 3$, the results remain the same. Inequality (2.1) for $N = \langle 3, m, l\rangle$ holds iff

$$m^2 + l^2 + 9m + 9l - 4ml - 24 > 0. \tag{2.4}$$

Note $l$ can only take values from the gaps of $\langle 3, m\rangle$. By [Rød94] the largest gap of $\langle 3, m\rangle$ is $2m-3$. Hence

$$m + 1 \le l \le 2m - 3.$$

In particular, $m = 4$ forces $l = 5$, and $m = 5$ forces $l = 7$; one checks that both satisfy Inequality (2.4).

It remains to consider $m \ge 7$, where we apply some basic calculus. Let $f(m,l)$ denote the left-hand-side in equation (2.4), and evaluate at $l = m+1$,

$$f(m, m+1) = -2(m-7)(m-1) < 0 \text{ for } m \ge 7.$$

Fixing $m$, the derivative of $f(m,l)$ with respect to $l$ is $2l+9-4m$, which is negative when $l < 2m-4.5$. In particular, this derivative is always negative for $l \in [m+1, 2m-3]$. Thus no numerical semigroup $\langle 3, m, l\rangle$ with $m \ge 7$ satisfies Inequality (1.3). □

Both $\langle 3,4,5\rangle$ and $\langle 3,5,7\rangle$ are non-symmetric semigroups. In fact,

**Corollary 2.16.** *Suppose $N = \langle 3, m, l\rangle$, where $3 < m < l$, then $N$ is not symmetric.*

*Proof.* Suppose not. By Lemma 2.7, the largest gap $b_g$ of $\langle 3, m, l\rangle$ is extremal. Recall $b_g = l - 3$ and $g = \frac{m+l-3}{3}$ from the preceding proof. The equality $b_g = 2g-1$ then becomes $l = 2m$, which contradicts $l \le 2m-3$. □

We conclude, as in Theorem 2.3, the numerical semigroups that satisfy (1.3) are

(1) *All hyperelliptic semigroups*
(2) *Symmetric:* $\langle 3,4\rangle, \langle 3,5\rangle, \langle 3,7\rangle, \langle 4,5,6\rangle$.
(3) *Non-symmetric:* $\langle 3,4,5\rangle$, $\langle 3,5,7\rangle$.

## References


[AFS14] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth, *Singularities with* $\mathbb{G}_m$*-action and the log minimal model program for* $\overline{M}_g$, 2014, To appear in *Journal für die Reine und Angewandte Mathematik*, DOI: 10.1515/crelle-2014-0063.

[Rød94] Øystein J. Rødseth, *A note on T. C. Brown and P. J.-S. Shiue's paper: "A remark related to the Frobenius problem" [Fibonacci Quart.* **31** *(1993), no. 1, 32–36; MR1202340 (93k:11018)]*, Fibonacci Quart. **32** (1994), no. 5, 407–408. MR 1300276